Quantitative Finance

Economics, Finance and Management

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Example (Payments in Increasing Arithmetic form)

O John is buying a computer and iPhone payable in three instalments: €950, 1000 e 1050 (interest included, 9% annual).Present Value?

$$\begin{aligned} P.V. &= 950.00(1.09)^{-1} + 1000.00(1.09)^{-2} + 1050.00(1.09)^{-3} \\ &= 2524.03€ \end{aligned}$$

$$\begin{array}{llllll} = & 950(1.09)^{-1} & + & 1000(1.09)^{-2} & + & 1050(1.09)^{-3} \\ \hline = & 900(1.09)^{-1} & + & 900(1.09)^{-2} & + & 900(1.09)^{-3} \\ + & 50(1.09)^{-1} & + & 50(1.09)^{-2} & + & 50(1.09)^{-3} \\ & & + & 50(1,09)^{-2} & + & 50(1.09)^{-3} \\ & & & & + & 50(1.09)^{-3} \\ \hline = & 900 a_{\overline{3|}9\%} & + & & & \\ + & 50 a_{\overline{3|}9\%} & + & 50_{1|} a_{\overline{2|}9\%} & + & 50_{2|} a_{\overline{1|}9\%} \\ \hline = & \multicolumn{5}{l}{€2524.03 = (950-50) a_{\overline{3|}} + 50\,(Ia)_{\overline{3|}}} \end{array}$$

Result (P.V. Payments in Increasing Arithmetic form)

Present Value (P.V.)

$$\begin{aligned} PV &= (C-h)\, a_{\overline{n|}} + h\left(a_{\overline{n|}} + {}_{1|}\, a_{\overline{n-1|}} + {}_{2|}\, a_{\overline{n-2|}} + \ldots + {}_{n-1|}\, a_{\overline{1|}}\right) = \\ &= (C-h)\, a_{\overline{n|}i} + \mathbf{h}.\, (\mathbf{Ia})_{\overline{n|}\mathbf{i}} \end{aligned}$$

where, simplifying,

$$(Ia)_{\overline{n|}i} = \frac{\ddot{a}_{\overline{n|}i} - nv^n}{i} = \frac{\ddot{a}_{\overline{n|}i} - n(1+i)^{-n}}{i}$$

- *If* $C = h$ *then* $PV = h \cdot (Ia)_{\overline{n|}i}$
- *If* $C = h$ *and* $h = 1$ *then* $PV = (Ia)_{\overline{n|}i}$

Example (Payments in Decreasing Arithmetic form)

Edward is buying an *iPod* payble in 4 monthly instalments of 45, 35, 25 e 15€. $i_M = 0,012$, $h^* = -10$. P.V.?

$$PV = 45(1.012)^{-1} + 35(1.012)^{-2} + 25(1.012)^{-3} + 15(1.012)^{-4}$$

Looking backwards,

$$\begin{array}{lllll}
PV = & 45v & +35v^2 & +25v^3 & +15v^4 \\
\hline
= & 5v & +5v^2 & +5v^3 & +5v^4 \\
+ & 10v & +10v^2 & +10v^3 & +10v^4 \\
+ & 10v & +10v^2 & +10v^3 & \\
+ & 10v & +10v^2 & & \\
+ & 10v & & & \\
\hline
= & 5a_{\overline{4}|} & & & \\
+ & 10\left(a_{\overline{4}|}\right. & +a_{\overline{3}|} & +a_{\overline{2}|} & \left.+a_{\overline{1}|}\right) \\
\hline
= & (15-10)\,a_{\overline{4}|} + 10\,(Da)_{\overline{4}|} & & & \\
= & 5a_{\overline{4}|} + 10\frac{4-a_{\overline{4}|1,2\%}}{0,012} & & &
\end{array}$$

Result (P.V. Payments in Decreasing Arithmetic form, last pay D)

Last Payment D, look backwards,

$$PV = (D - h)\, a_{\overline{n|}i} + h \cdot (Da)_{\overline{n|}i}$$

$$
\begin{array}{rcl}
(Da)_{\overline{n|}i} & = & nv + (n-1)v^2 + (n-2)\,v^3 + \ldots + 2v^{n-1} + v^n \\
\hline
 & & \begin{array}{cccccccc}
 +v & +v^2 & +v^3 & +\ldots & +v^{n-1} & +v^n & + \\
 +v & +v^2 & +v^3 & +\ldots & +v^{n-1} & & + \\
 +v & +v^2 & +v^3 & +\ldots & & & + \\
 & & & \ldots & \ldots & \ldots & + \\
 +v & +v^2 & & & & & + \\
 +v & & & & & & =
 \end{array} \\
\hline
 & = & a_{\overline{n|}i} + a_{\overline{n-1|}i} + \ldots + a_{\overline{3|}i} + a_{\overline{2|}i} + a_{\overline{1|}i} \\
(Da)_{\overline{n|}i} & = & \dfrac{n - a_{\overline{n|}i}}{i}, \text{ P.V. with } D = h = 1)
\end{array}
$$

Example

John is buying a computer, paying in three instalments: 1st €950, others with 25% increase, $i = 9\%$.

$$\begin{aligned} PV &= 950(1,09)^{-1} + 950 \times 1.25(1,09)^{-2} + 950 \times 1.25^2(1,09)^{-3} \\ &= 3017,26€ \end{aligned}$$

0	1	2	3	$\cdots$	$n-1$	n
	P	$P\,h$	$P\,h^2$	$\cdots$	$P\,h^{n-2}$	$P\,h^{n-1}$

- 1st Payment: C; rate: $r = h$.
- If $h > 1$, annuity is increasing;
- If $0 < h < 1$, annuity is decreasing.
- P.V.: Geometric series with rate hv:

$$\begin{aligned}
\text{P.V.} &= Pv + Phv^2 + Ph^2v^3 + \ldots + Ph^{n-1}v^n \\
&= P\left(\frac{v - h^{n-1}v^n \times hv}{1 - hv}\right) = P\left(\frac{v\,(1 - h^n v^n)}{v\,(1/v - h)}\right) \\
&= P\left(\frac{1 - h^n\,(1+i)^{-n}}{1 + i - h}\right)
\end{aligned}$$